A miscellany of basic issues on incompressible fluid equations

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Abstract. We consider basic issues of incompressible fluid dynamical equations.

1. Introduction
This article describes some open problems regarding the basic issues on the incompressible fluid dynamics, followed by some idiosyncratic remarks.

We will be interested in the incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0$$

and also in the incompressible Euler equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \quad \nabla \cdot u = 0.$$

Both of them still pose formidable mathematical challenges in spite of the efforts that have made for long time. We consider fluid flows in $\mathbb{R}^n$ (theoretically) or $\mathbb{T}^n$ (numerically), with $n = 2, 3$. In recent years there have been a number of more complete and comprehensive surveys published on the similar subject, say, [1, 2, 3, 4, 5, 6, 7, 8, 9] to name just a few. It is not our intention to give yet another survey with an extensive list of publications, but rather to give an informal introduction geared particularly for non-specialists. For the most part, this paper includes on-going materials which are rather preliminary. Hopefully some of the ideas may find their ways to future developments and they will not turn out to be environmental pollution.

We discuss the Euler equations in Chapter 2 and the Navier-Stokes equations in Chapter 3. In Chapter 4, a summary is given together with miscellaneous remarks.
2. Euler equations

2.1. BKM and BMO

Let us consider the celebrated BKM (Beale-Kato-Majda) theorem [10], which states a possible onset of singularity in Euler flows must show up in vorticity.

In [11], Pumir and Siggia noted, with uncanny skill, that “Thus essential property to explain is why the strain is predominantly perpendicular to vorticity. The vorticity at a point does not generate parallel strain, as is apparent from the integral
\[
\frac{d^3}{d^3} \rho \omega(r) \land \omega(r + \rho) - 3 \rho \land \omega(r + \rho) \rho \cdot \omega(r)
\]
\[
= \int \frac{d^3}{d^3} \rho \omega(r) \land \omega(r + \rho) - 3 \rho \land \omega(r + \rho) \rho \cdot \omega(r)
\]
In the vicinity of \( \rho = 0 \) and after doing an angular average, the numerator vanishes as \( \rho^2 \). Hence with no additional assumptions better estimates should be possible for \( \omega \cdot \epsilon \cdot \omega / \omega^2 \) than in Ref.32.” Here, Ref.32 means the BKM paper.

We can now judge the accuracy of their prophecy. At the level of the BKM theorem which deals with a sup norm of the vorticity, it was not possible to rule out a possibility that vorticity blows up mildly as a logarithmic function of space variables. According to recent updates using a BMO (Bounded Mean Oscillations) norm e.g. [12], we can safely exclude such possibilities because BMO is a function space where weakly singular functions, such as logarithmic functions, reside. (For other generalisations and variants of BKM theorem, see e.g. [13].)

This progress reminds us of an ODE-PDE analogy in the two different kinds of regularity problems of differential equations, which is due to Yudovich’s observation [14]. Corresponding to the Euler equations, we have a system of ordinary differential equations
\[
\frac{dx}{dt} = u(x, t), \quad x(0) = a,
\]
which defines particle trajectories \( x(t) \) subject to the Euler flow \( u(x, t) \), starting from initial positions \( a \). It is well-known that Lipshitz condition
\[
\delta u(r) \leq Kr
\]
is sufficient to ensure unique existence of solution to (3). Here \( \delta u(r) \equiv |u(x, t) - u(y, t)| \)
\[
r \equiv |x - y| \quad \text{and} \quad K \quad \text{is a constant.}
\]
This, however, is not necessary. A necessary and sufficient condition is known as:
\[
\int_{r>0} \frac{dr}{\delta u(r)} = \infty.
\]
For example, the latter condition is satisfied with \( \delta u(r) \propto r \log r \), often referred to as Osgood, or quasi-Lipshitz condition, see e.g. [15, 16]. The latter corresponds to vorticity \( \propto \log r \), which in turn corresponds to the BMO criterion. \( \dagger \)

\( \dagger \) Also possible are \( \delta u(r) \propto r \log r (\log r)^2 \), \( \delta u(r) \propto r \log r (\log r)^2 (\log r)^3 \), and so on.
condition of the Euler equations (PDE) seems to be in parallel with that of the equations of trajectories (ODE). This is the Yudovich’s observation.

It should be noted that we have alternative forms of the necessary and sufficient condition for the regularity of the solutions to (3), We will take a brief look at one of them, so-called Okamura theory, [17, 18, 19]. The lesser known condition sends the regularity problem of ODEs into that of a linear partial differential inequality. There, a scalar quantity called, Okamura function, a variant of Lyapunov function, plays a key role. It may be regarded as a two-point version of passively advected scalar, i.e. a scalar advected by velocity at two different points in space. Potentially, it offers a place for interpreting the ODE-PDE analogy on this basis.

We consider 2D Euler equations and point out similarities and differences between the Okamura function and the stream function. We also compare with dynamics of a usual passive scalar.

**Theorem** We assume that \( u(x, t) \) is continuous on an \((n + 1)\)-dimensional space-time region \( D \). The necessary and sufficient conditions that we have a unique solution from \( D \) is given by the following.

There is some \( (t; y, z) \) which is \( C^1 \) for \((y, t) \in D, (z, t) \in D\) such that
\[
\Phi(t, y, z) = 0 \text{ if } y = z,
\]
\[
\Phi(t, y, z) > 0 \text{ if } y \neq z,
\]
and
\[
D^* \Phi \equiv \frac{\partial \Phi}{\partial t} + u_i(y, t) \frac{\partial \Phi}{\partial y_i} + u_i(z, t) \frac{\partial \Phi}{\partial z_i} \leq 0.
\]

The initial condition is given by \( \Phi(0, y, z) = |y - z| \). We consider spatial dimensions \( n = 2 \) or \( 3 \). Note that the RHS of (7) is obtained by the time derivative of the form \( \frac{D}{dt} \Phi(t, y(t), z(t)) \) by chain rule. See Appendix for an intuitive understanding of the theorem. (Note that \( \Psi = D(P, Q) \) and \( t = t^Q - t^P \) therein.)

The partial differential inequality is interesting and insightful as it may be regarded as an equation for a passive scalar advected by a flow at two distant spatial points. Since the properties of the inequality is not well understood, it may be in order to check how \( D^* \Phi \) behaves for a number of choices of \( \Phi \).

For steady 2D incompressible flows, the Okamura function may be related with a stream function. To see this, let us recall some basic facts as follows.

- For a fixed time, the stream function \( \psi \) in a 2D incompressible flow decides whether or not two spatial points lie on a single streamline, as it measures volume flux across the segment ending at the two points.
- For two different space-times, the Okamura function decides whether or not two spatial points lie on a single particle path.

\[\text{§ The asterisk } * \text{ is used here merely to avoid confusion with a standard Lagrangian derivative.}\]
• For steady flows $\partial/\partial t \equiv 0$, particle paths coincide with streamlines. (For unsteady flows, particle paths are the envelopes of streamlines.)

To illustrate the meaning of the above formalism, let us consider a special (rather trivial) case where $\Phi$ has the form

$$\Phi(t, y, z) = \phi(y, t) - \phi(z, t).$$

(8)

This choice actually deviates from the original formulation, as it is not a norm, that is, it is not positive-definite.

We consider steady cases by setting $\phi(x) = \psi(x)$ in the above and we have

$$D^*\Phi = u_i(y) \frac{\partial \psi}{\partial y_i} - u_i(z) \frac{\partial \psi}{\partial z_i} = 0.$$

Here use has been made of the identity $u \cdot \nabla \psi = 0$ which follows from the definition of a stream function $u = \nabla^\perp \psi = (\partial_y \psi, -\partial_x \psi)^T$. Hence, the equality holds in (7). In this sense the Okamura function is related to the steady stream function in 2D.

It would be interesting to explore an analogy in more general unsteady cases. Assuming that the velocity gradient is finite (of course, the Lipshitz condition is then satisfied).

We have $\Phi \approx l \cdot \nabla \phi$ to leading order and

$$D^*\Phi \approx \frac{\partial \Phi}{\partial t} + (u \cdot \nabla)\Phi + (l \cdot \nabla u_i) \frac{\partial \phi}{\partial x_i}$$

$$= \frac{D\Phi}{Dt} + ((l \cdot \nabla)u) \cdot (\nabla \phi).$$

Here $l$ is a material element because both of its end points $y$ and $z$ keep their particle trajectories, that is, it satisfies

$$\frac{DL}{Dt} = (l \cdot \nabla)u.$$

We find

$$\frac{D\Phi}{Dt} = (l \cdot \nabla u_i) \frac{\partial \phi}{\partial x_i} + l \cdot \frac{D\phi}{Dt} \nabla \phi,$$

or, $= (l \cdot \nabla) \frac{D\phi}{Dt}$.  

(9)

Up to here $\phi$ is left undetermined.

**Case (i)**

If $\phi$ is a passive scalar, that is,

$$\frac{D\phi}{Dt} = 0$$

we find

$$D^*\Phi = (l \cdot \nabla u) \cdot (\nabla \phi)$$

$$= \left( S \cdot l + \frac{1}{2} \omega \times l \right) \cdot (\nabla \phi)$$

Care should be taken in that in the mathematical literature sometimes the term 'streamlines' is used to mean 'particle paths' (or, 'particle trajectories').
where we have chosen \( l = \omega \) in the last line. We recall experimental observations on alignments that

\[ S \cdot l \approx \lambda_2 l, \]

\[ S \cdot (\nabla \phi) \approx \lambda_3 (\nabla \phi), \]

where \( \lambda_i \) (\( i = 1, 2, 3 \)) are the eigenvalues of the rate-of-strain tensor \( S \) ordered as \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \), (\( \lambda_1 + \lambda_2 + \lambda_3 = 0 \)). Then we have \( D^* \Phi \approx 0 \).

**Case (ii)**

Take \( \phi(\mathbf{x}, t) = \psi(\mathbf{x}, t) \) in 2D. By (9), we have

\[
D^* \Phi \approx 2((l \cdot \nabla) \mathbf{u}) \cdot (\nabla \psi) + l \times \nabla p
\]

\[
= 2\mathbf{u} \times ((l \cdot \nabla) \mathbf{u}) + l \times \nabla p.
\]

Here use has been made of \( \nabla \psi = -\mathbf{u}^\perp \) and the 2D Euler equations in the form

\[ \frac{D}{Dt} \nabla \psi = \nabla^\perp p. \]

Alternatively, a more general expression for \( D^* \Phi \) can be obtained for \( \Phi(\mathbf{y}, \mathbf{z}, t) = \psi(\mathbf{y}, t) - \psi(\mathbf{z}, t) \) as follows

\[
D^* \Phi = \frac{\partial \psi}{\partial t}(\mathbf{x} + l, t) - \frac{\partial \psi}{\partial t}(\mathbf{x}, t),
\]

where we have put \( \mathbf{y} = \mathbf{x} + l, \mathbf{z} = \mathbf{x} \).

By a standard method, the equation for stream function in 2D Euler dynamics can be derived, using (29) below, as

\[
\frac{\partial \psi}{\partial t}(\mathbf{x}, t) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}^2} \{(\mathbf{x} - \mathbf{y}) \times \nabla \psi\} \{(\mathbf{x} - \mathbf{y}) \cdot \nabla \psi\} \frac{dy}{|\mathbf{x} - \mathbf{y}|^2}.
\]

Note that \( \frac{\partial \psi}{\partial t}(\mathbf{x}, t) = \frac{D \psi}{Dt}(\mathbf{x}, t) \).

For infinitesimal \( l, \Phi \approx l \cdot \nabla \psi = \mathbf{u} \times l \) is volume flux across \( l \) and

\[
D^* \Phi \approx \frac{\partial}{\partial t} l \cdot \nabla \psi
\]

\[
= \frac{D}{Dt}(l \cdot \nabla \psi) - (u \cdot \nabla)(l \cdot \nabla \psi)
\]

\[
= (l \cdot \nabla) \frac{D \psi}{Dt} - (u \cdot \nabla)(u \times l).
\]

If \( D^* \phi < 0 \) at \( t = 0 \), we may reverse its sign by \( l \rightarrow -l \) because \( D^* \phi \) is linear in \( l \). The problem is then whether and how we may make \( D^* \Phi \approx 0 \), under the evolution of \( \psi \). At present a connection of the stream function to the Okamura function in unsteady cases is obscure.
2.2. Use of other gauges

Consider fluid flows in \( \mathbb{R}^3 \) or \( T^3 \). We may use either velocity or vorticity for its description. We may take yet another choice, impulse variable for the same purpose e.g. [20]. The impulse variable is by definition an un-curl of vorticity

\[ \gamma = (\nabla \times)^{-1} \mathbf{\omega} = \mathbf{u} + \nabla \phi, \]

which is governed by

\[ \frac{D\gamma}{Dt} = - (\nabla \mathbf{u})^T \gamma + \nabla \left( \frac{D\phi}{Dt} + \frac{|\mathbf{u}|^2}{2} - p \right), \tag{10} \]

or, equivalently, by

\[ \frac{\partial \gamma}{\partial t} = \mathbf{u} \times \mathbf{\omega} + \nabla \left( \frac{\partial \phi}{\partial t} - \frac{|\mathbf{u}|^2}{2} - p \right), \tag{11} \]

where \( \lambda = \Lambda + \mathbf{u} \cdot \gamma \) and

\[ \phi = \int_0^t \left( p - \frac{|\mathbf{u}|^2}{2} \right) \text{fixed } \mathbf{a} \ dt' + \int_0^t \lambda dt' = \psi - \Phi. \tag{12} \]

Here, we have defined

\[ \psi \equiv \int_0^t \left( p - \frac{|\mathbf{u}|^2}{2} \right) \text{fixed } \mathbf{a} \ dt', \quad \Phi \equiv - \int_0^t \lambda dt'. \]

For the general choice of gauge, Weber transform takes the form

\[ \gamma_j \frac{\partial x_j}{\partial a_i} - \gamma_i(0) = \frac{\partial}{\partial a_i} \int_0^t \lambda(t') dt' = - \frac{\partial \Phi}{\partial a_i} \tag{13} \]

and Cauchy formula associated with it is given by

\[ \gamma_i(t) = \left( \gamma_j(0) - \frac{\partial \Phi}{\partial a_j} \right) \frac{\partial a_j}{\partial x_i} = \gamma_j(0) \frac{\partial a_j}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}. \tag{14} \]

By (14), the condition for regular evolution in \([0, T]\) is given by

\[ \int_0^T \| \gamma + \nabla \Phi \|_\infty^2 dt < \infty, \]

which is the same as

\[ \int_0^T \| \mathbf{u} + \nabla \psi \|_\infty^2 dt < \infty \]

because of \( \gamma + \nabla \Phi = \gamma + \nabla (\psi - \phi) = \mathbf{u} + \nabla \psi \) (impulse in geometric gauge). The choice \( \lambda = 0 \) (\( \Lambda = - \mathbf{u} \cdot \gamma \)) corresponds to geometric gauge. Note that \( \Lambda = 0 \) (\( \lambda = \mathbf{u} \cdot \gamma \)) corresponds to zero gauge.
We will particularly consider the choice of zero gauge, because the impulse equation in this case is rather special in that it has no stretching terms [20]. In geometric gauge, we have a criterion counterpart to the BKM result [21]:

$$\int_0^T \| \gamma \|_\infty^2 dt < \infty, \quad (15)$$

which is mentioned above.

One question is can we obtain a similar criterion using impulse variables in other choices of gauge? In particular, can the impulse variable in zero gauge remain bounded, even at the onset of a putative singularity formation?

In this case the governing equations read

$$\frac{\partial \gamma}{\partial t} = u \times (\nabla \times \gamma) = u \times \omega, \quad (16)$$

or, equivalently,

$$\frac{D \gamma_i}{Dt} = u_j \frac{\partial \gamma_i}{\partial x_j} = -\gamma_j \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_i} (\gamma_j u_j). \quad (17)$$

Now, we consider how $|\gamma|$ grows in this gauge:

$$\frac{\partial}{\partial t} \frac{|\gamma|^2}{2} = \gamma \cdot (u \times \omega) \quad (18)$$

$$= \nabla \phi \cdot (u \times \omega) \quad (19)$$

$$= \omega \cdot (\nabla \phi \times \gamma). \quad (20)$$

In the first, second and third form, $\nabla \phi$, $\gamma$ and $u$ have been eliminated, respectively (recall $u = \gamma - \nabla \phi$). A kind of nonlinear depletion can be observed most clearly in the third form, under the assumption that

$$|u| \ll |\gamma|, |\nabla \phi|,$$

which is observed numerical simulations using the geometric gauge.

Clearly we have

$$\frac{\partial}{\partial t} \log |\gamma| \leq |\omega| \frac{|u|}{|\gamma|}.\quad (21)$$

If we assume that

$$\frac{|u|}{|\gamma|} \leq K, \quad (K = \text{constant})$$

then, at every point in space we have

$$|\gamma(t)| \leq |\gamma(0)| \exp \left( K \int_0^t \| \omega(t') \|_\infty dt' \right).$$

With zero gauge, this says that $|\gamma|$ may remain bounded at the possible onset of singularity. Numerical simulations of Euler flows are used to compare impulse in geometric and zero gauges. We use $256^3$ grid points for $2/3$-dealiased Fourier spectral...
method together with forth-order Runge-Kutta method. The initial condition has an energy spectrum

\[ E(k) = ck^2 \exp(-k^2), \]

where \( c \) is fixed such that the total entropy normalised: \( \frac{1}{2} \langle |\omega|^2 \rangle = 1 \). We show in Fig.1 the time evolution of

\[ E_\gamma(t) = \frac{1}{2} \langle |\gamma|^2 \rangle, \]

for two different choices of gauge. We see on a time interval over which flows are resolved that

\[ \langle |\gamma_{\text{zero}}|^2 \rangle \ll \langle |\gamma_{\text{geom}}|^2 \rangle, \]

where \( \gamma_{\text{zero}} \) and \( \gamma_{\text{geom}} \) denote impulse variables in each gauge. If an Euler flow goes singular, then we must have \( |\gamma_{\text{geom}}| \to \infty \) at some points in space, but what about \( |\gamma_{\text{zero}}| \)?

Is it the case that the possible onset of singularity can be monitored only with the geometric gauge and and is invisible to other choices of gauge? Physical intuition may suggest that a mere change of the gauge should not alter drastically what it can detect. However, noting that the impulse variable reduces to the usual incompressible velocity as a special case and that no criteria for the Euler singularity are known in terms of velocity, this is not a trivial problem.

3. Navier-Stokes equations

The two seminal papers are [22] and [23]. A useful guide for the latter is a commentary by Serrin in [24]. These papers have established global in time existence of “weak solutions” for the Navier-Stokes equations, but have not been successful in showing the existence in classical sense (i.e. as a smooth function). Hence, “E. Hopf proved that there exists at least one 'weak' solution 'globally' to the initial-boundary problem....
Figure 2. The ratio $|R(\eta)|$ (solid) and $Ku(\eta)^2$ (dashed) in the Burgers vortex, where $K = 8\pi^2\nu/(5\alpha \Gamma^2)$ is a constant.

Figure 3. The ratio $|r|$ (solid) and $10|u|^2$ (dashed) in Numerical simulation. Both are taken at the point of peak velocity.

However, as we will see in Chap.6 Sec.7, Hopf’s class of ‘weak’ solutions is intolerably wide. That is, uniqueness can be broken there.” [25]

The original physical motivation for these basic problems may be as follows. If the assumptions based on classical physics employed in the derivation of the Navier-Stokes equations are correct, the physicists believe that there should be a (unique) solution all times, which would in turn justify the assumptions introduced therein. In other words, to argue the existence and uniqueness of solutions to the Navier-Stokes equations helps to double check the plausibility of physical considerations used to derive them.
3.1. Lack of Maximum principle

A regularity criterion for the Navier-Stokes equations is given by the maximum value of the velocity. Alternatively, it is given by a spatial integral \( Q(t) \), called enstrophy:

\[
Q(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\omega|^2 dx,
\]

see e.g. [7, 26, 27]. It is well-known that a standard analysis leads to a differential inequality for \( Q(t) \) of the following form:

\[
\frac{dQ(t)}{dt} \leq c \nu \ Q(t)^3,
\]

where \( c \) is a constant. It ensures that \( Q(t) \) remains finite, but does so only for a short time.

It may help to understand the difficulty by comparing [27] with the 3D Burgers equations

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u, \tag{21}
\]

which is known to be regular all time. It has no pressure. In terms of \( Q(t) \), we would get stuck in an attempt to prove the boundedness of \( Q(t) \) for a long time, as we did in the case of Navier-Stokes equations. This is clear since both equations have the same dimensional structure.

Then how can we prove the global regularity for the Burgers equations? In fact, it is proved using a maximum principle [28]. Let’s see how it fails for the Navier-Stokes equations.

In (1), by dotting with \( u \), we find an equation for energy density

\[
\frac{\partial |u|^2}{\partial t} + (u \cdot \nabla) |u|^2 = -u \cdot \nabla p + \nu u \Delta u. \tag{22}
\]

At the points of peak velocity, we have

\[
\frac{\partial |u|^2}{\partial t} = -u \cdot \nabla p - \nu |\nabla u|^2 + \nu |u|^2, \tag{23}
\]

hence we find

\[
\frac{\partial |u|^2}{\partial t} \leq -u \cdot \nabla p + \nu \Delta |u|^2. \tag{24}
\]

In the case of pressure-less Burgers equations, we would have instead

\[
\frac{\partial |u|^2}{\partial t} \leq \nu \Delta |u|^2,
\]

from which we conclude by applying the maximum principle

\[
\max_x \frac{|u(t)|^2}{2} \leq \max_x \frac{|u(0)|^2}{2},
\]

\[\text{\textbullet} \quad \text{For a class of potential flows } u = \nabla \phi \text{ we may use Forsyth-Hopf-Cole transform to linearise it, thereby proving its regularity directly.} \]
and the regularity follows.

In the Navier-Stokes case, there is no reason to expect that $\mathbf{u} \cdot \nabla p > 0$. Therefore, it may be useful to ask under what conditions the following inequality holds $|\mathbf{u} \cdot \nabla p| \leq \nu |\nabla \mathbf{u}|$. + In other words, in terms of

$$R(t) \equiv \frac{\mathbf{u} \cdot \nabla p}{\nu |\nabla \mathbf{u}|^2},$$

when and where does $|R(t)| \leq 1$ hold? It is too optimistic to expect it holds generally. But because apparently there have been no previous reports on the issue, we consider it by a model and by numerical simulations.

First we consider a model, the Burgers vortex which is an exact solution of the Navier-Stokes equations. It is known that there are similar tubular structure in fully developed turbulence. The solution has the form

$$\mathbf{u} = (-\alpha x + u(x, y, t), -\alpha y + v(x, y, t), 2\alpha z).$$

In cylindrical polar coordinates, axial vorticity is given by

$$\omega = \frac{\alpha \Gamma}{2\pi \nu} \exp\left(-\frac{\alpha r^2}{2\nu}\right),$$

and the velocity components by

$$u_{\theta} = \frac{\Gamma}{2\pi r} \left(1 - \exp\left(-\frac{\alpha r^2}{2\nu}\right)\right), \quad u_r = -\alpha r, \quad u_z = 2\alpha z.$$

In this solution, the pressure is given explicitly by

$$p = -\frac{1}{2} \alpha^2 (r^2 + 4z^2) + \int_{0}^{r} \frac{u_{\theta}(s)^2}{s} ds.$$

Using those expressions we may evaluate the two terms as

$$\mathbf{u} \cdot \nabla p = \alpha^3 (r^2 - 8z^2) - \alpha u_{\theta}(r)^2,$$

and

$$\nu |\nabla \mathbf{u}|^2 = 6\nu \alpha^2 + \nu \left(\frac{u_{\theta}(r)}{r}\right)^2 + \left(\frac{\partial u_{\theta}(r)}{\partial r}\right)^2.$$

While the velocity components are decoupled in this solution, but it is of interest to check how $\mathbf{u} \cdot \nabla p$ behaves. Taking only the contribution by perturbations which do not depend on $r$ and $z$ explicitly, that is, the final term on the RHS of (25, 26), we

+ It should be noted a huge velocity gradient (related to “dissipation anomaly”) would make it easier for the inequality to hold. Note also that in the initial, almost inviscid stage the RHS is small and that the inequality is unlikely to hold.
notice that $\mathbf{u} \cdot \nabla p \leq 0$, that is, it has a wrong sign to make the maximum principle work. However, if we normalise it as $R(r)$, we find, again taking the fluctuations only

$$|R(t)| = 2 \frac{(1-e^{-\eta})^2}{\left(2e^{-\eta} - \frac{1-e^{-\eta}}{\eta}\right)^2 + \left(\frac{1-e^{-\eta}}{\eta}\right)^2}, \quad \eta \equiv \frac{\alpha r^2}{2\nu},$$

where $\eta$ is a non-dimensional radius. This is plotted in Fig.1, together with a measure of the squared velocity $u_2^2$, that is, $(1 - \exp(-\eta))^2/(5\eta)$. In the case of the Burgers vortex, the ratio is smaller than 1 everywhere. The maximum velocity is a guide of a radius of the vortex. This suggests a possibility that such an inequality holds once tubular vortex structures form and dominate the flow field. In this sense, vortex tubes are favorable structure for the maximum principle to apply.

Next we consider a result from numerical simulations in Fig.2. Unfortunately, $|\mathbf{u} \cdot \nabla p|$ taken at the point of peak velocity is large and negative $R(t) \ll -1$ in the early stage of the development where vortex layers dominate the flow field. Later $|R(t)|$ decreases in time, as expected. Note that the enstrophy attains its maximum at $t = 8$, by that time vortex tubes are dominant. This suggests a breakdown of the maximum principle, but also leaves a possibility of better controlling of $|\mathbf{u} \cdot \nabla p|$ after tubular vortices are formed. (Of course, that would be too late for arguing regularity anyway. But it may be of use to know what is actually happening.)

Finally, we take a closer look at the term $\mathbf{u} \cdot \nabla p$ using singular integrals explicitly. Consider Poisson’s formula

$$\phi(x) = -\frac{1}{4\pi} \int \frac{f(y)dy}{|x-y|}$$

for $\Delta \phi(x) = f(x)$. We have by Carderon’s formula

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{f(x)}{3} \delta_{ij} + T_{ij}[f](x)$$

where

$$T_{ij}[f](x) = \frac{1}{4\pi} \text{PV} \int \left( \frac{\delta_{ij}}{|x-y|^3} - \frac{3(x_i - y_i)(x_j - y_j)}{|x-y|^5} \right) f(y)dy$$

is a convolution against a dipole kernel.

By taking each component of $u_i u_j$ as $f$, we find

$$p = -\frac{|\mathbf{u}|^2}{3} - T_{ij}[u_i u_j],$$

and hence

$$\nabla p = -\nabla \frac{|\mathbf{u}|^2}{3} - \nabla T_{ij}[u_i u_j].$$

Only the second term in the above matters. By subtracting

$$\frac{1}{4\pi} \text{PV} \int \left( \frac{\delta_{ij}}{|x-y|^3} - \frac{3(x_i - y_i)(x_j - y_j)}{|x-y|^5} \right) u_i(x)u_j(x)dy = 0$$
we obtain
\[ T_{ij}[u_i u_j] = \frac{1}{4\pi} \text{PV} \int \left( \frac{\delta_{ij} - 3(x_i - y_i)(x_j - y_j)}{|x - y|^3} \right) (u_i(y)u_j(y) - u_i(x)u_j(x)) dy \]

It is easy to show
\[ \frac{\partial}{\partial x_k} T_{ij}[u_i u_j] = -\frac{3}{4\pi} \text{PV} \int \left( \frac{\delta_{ij} r_k + \delta_{jk} r_i + \delta_{ki} r_j}{r^5} - 5 \frac{r_i r_j r_k}{r^7} \right) (u_i' u_j' - u_i u_j) dy, \]
where \( r = x - y \) and \( u_i = u_i(x, t) \), \( u_i' = u_i(y, t) \). We then find
\[ u \cdot \nabla T_{ij}[u_i u_j] = -\frac{3}{4\pi} \text{PV} \int ((e \cdot u)|u'|^2 - 3(e \cdot u)|u|^2 + 2(e \cdot u')(u \cdot u') - 5(e \cdot u)(e \cdot u')^2 + 5(e \cdot u)^3) \frac{dy}{r^3}, \]
where \( e = r/|r| \). No depletion mechanism is explicitly observed here. It may worth handling singular integral transforms explicitly, e.g. [33], to get a better estimate, say under an assumption that the flow field is dominated by vortex tubes.

3.2. Zero gauge revisited

The breakdown of the maximum principle is due to the presence of the (nonlocal) pressure term \( p \). One way to eliminate it is to take other choices of gauge, other than using vorticity.

The Navier-Stokes equations written in the form
\[ \frac{\partial u}{\partial t} = u \times \omega - \nabla \left( p + \frac{|u|^2}{2} \right) + \nu \Delta u \]  
may be recast into impulse equations with zero gauge. Introducing
\[ \frac{\partial \phi}{\partial t} = p + \frac{|u|^2}{2} + \nu \Delta \phi \]  
and recalling \( \gamma = u + \nabla \phi \) we find
\[ \frac{\partial \gamma}{\partial t} = u \times \omega + \nu \Delta \gamma, \]
thereby absorbing the notoriously cumbersome term \( u \cdot \nabla p \) into the dependent variable. The price we have to pay is the need to cope with solenoidal projection in the nonlinear term \( u = P[\gamma] \). Analyses of the Navier-Stokes equations in this form may be worth trying. It may be of interest to compare it with a “model” system
\[ \frac{\partial \gamma}{\partial t} = \gamma \times \omega + \nu \Delta \gamma, \quad \text{(model)} \]
In this case the nonlinear terms do not contribute to the growth of \( |\gamma| \). The corresponding vorticity equations take the form
\[ \frac{\partial \omega}{\partial t} + (\gamma \cdot \nabla) \omega = (\omega \cdot \nabla) \gamma - (\nabla \cdot \gamma) \omega + \nu \Delta \omega, \quad \text{(model)} \]
where \( \nabla \cdot \gamma \neq 0 \).
3.3. Self-similar solutions

When a solution depends on a particular combination of independent variables only, we say that it has similarity. In the case of the Navier-Stokes equations, the parameter kinematic viscosity determines this combination.

Let us consider partial regularity regarding similar solutions of the Navier-Stokes equations. When Leray faced a serious difficulty in an attempt to prove global existence, he suspected that some solutions might blow up in finite time and tried to construct an example of singular solutions. Specifically he searched similar solutions of the following form [22].

\[ u(x, t) = \frac{2a}{\sqrt{t_* - t}} U\left(\frac{x}{\sqrt{2a(t_* - t)}}\right), \quad a > 0 \]  

(34)

By transformation of variables, it is readily derived that \( U \) satisfies the following scaled equations

\[ a (X \cdot \nabla_X U + U) + U \nabla_X U = -\nabla_X P + \Delta_X U, \]  

(35)

\[ \nabla_X \cdot U = 0, \]  

(36)

where

\[ X = \frac{x}{\sqrt{2a(t_* - t)}} \]

and \( P \) is a scaled pressure. Leray was not able to find a non-trivial example of the above equation.

By performing \( \int_{\mathbb{R}^3} dX \cdot U \) on (35), we find

\[ \frac{a}{2} \int |U|^2 dX = \nu \int |\nabla_X U|^2 dX \]

Now take \( a = \nu/2 \) for simplicity, rescale \( U \rightarrow U/\nu \) to eliminate \( \nu \). We have

\[ \frac{1}{2} (X \cdot \nabla_X U + U) + U \nabla_X U = -\nabla_X P + \Delta_X U. \]  

(37)

Rosen introduced a series of norms

\[ \alpha_n = \int_{\mathbb{R}^3} |(\nabla_X \times)^n U(X)|^2 dX, \quad n = 0, 1, 2, \ldots \]  

(38)

and derived the following inequality in an Appendix of [29]

\[ \frac{(\alpha_0 \alpha_2^n)^{1/2}}{n \alpha_n} \geq \left(\frac{\pi}{2\sqrt{3}}\right)^{1/2} \alpha_0^{-1/8} \alpha_2^{-3/8} > 0. \]  

(39)

(See [26] for more systematic and sophisticated treatment.) This inequality actually shows partial regularity because the initial conditions are sufficiently smooth in the sense that

\[ \frac{(\alpha_0 \alpha_2^n)^{1/2}}{n \alpha_n} \to 0, \]
then there are no blow-up similar solutions evolving from them. Take, for example, initial conditions with compactly supported Fourier transforms $\hat{u}(k, t) = \text{const}(|k| \leq k_{\text{max}}), \quad 0(|k| > k_{\text{max}})$ We then have

$$
\frac{\left(\alpha_0 \alpha_{2n}\right)^{1/2}}{n \alpha_n} \propto \frac{1}{\sqrt{n}} \to 0 \quad (n \to \infty),
$$

which is at variance with (39). Hence, no similar solutions blow up for such initial conditions.

By a more direct approach Nečas et al. proved that (37) has no blowup solutions, that is, they proved that if $U \in L^3(\mathbb{R}^3)$ then $U \equiv 0$ [30]. More generally, it was proved in [31] that if $U \in L^q$ with $q > 3$, $U \equiv 0$. While Rosen’s result is restricted, it may be worth being noted in that it predates [30] by almost 30 years.

Rosen was brave enough to go on one step further and conjectured in general (that is, without assuming self-similarity) the following. For a series of unscaled norms

$$
A_n(t) = \int_{\mathbb{R}^3} |(\nabla \times)^n u(x, t)|^2 dx,
$$

(40)

if we demand some additional assumptions on top of

$$
\frac{(A_0(0)A_{2n}(0))^{1/2}}{n A_n(0)} \to 0, \quad n = 0, 1, 2, \ldots,
$$

than we might be able to prove global existence. To the best knowledge of the author, there are no publications that attempted to prove or disprove his conjecture.

To bound the growth of enstrophy, usually vorticity equations are used. In this sense, the analysis is slightly dynamical as it refers to infinitesimal time evolution. On the other hand, the effects of finite time evolution may be important for the flow to remain regular. For example, by introducing $\chi \equiv \nabla \times \omega$ we have

$$
\frac{D\chi}{Dt} = (\chi \cdot \nabla)u + 2\nabla \times ((\omega \cdot \nabla)u) + \nu \Delta \chi.
$$
We find

$$
\frac{d^2}{dt^2}\left(\frac{\omega}{2}\right) = \langle \omega \cdot S \cdot S \cdot \omega \rangle - \langle \omega \cdot P \cdot \omega \rangle + 2\nu \langle \Delta \omega \cdot S \cdot \omega \rangle + \nu \langle \omega \cdot \Delta S \cdot \omega \rangle - \nu \langle \chi \cdot S \cdot \chi \rangle - 4\nu \langle \nabla \times \chi \rangle \cdot \langle (\omega \cdot \nabla)u \rangle + 2\nu^2 \langle \nabla \times \chi \rangle^2,
$$

where the brackets denote spatial integral. While, already at a second-order in time, the viscous contribution is rather messy and not negative-definite at least at a glance, it may be worthwhile pursuing the line further.

4. Summary

In this paper, we have considered a number of basic problems regarding incompressible fluid mechanics.
For the Euler equations, similarities and differences between the Okamura function have been noted in connection with the ODE-PDE analogy. This suggests a difficulty of finding better estimates than those using BMO norms. Also, discussed is a special choice of gauge (zero gauge) in the impulse formulation.

For the Navier-Stokes equations, we have discussed the breakdown of the maximum principle for them. It is the term $u \cdot \nabla p$ which spoils the maximum principle argument. In a model of Burgers vortex, this is found to be small (though with an opposite sign). It is found numerically to be large negative in the developing stage of turbulence. This suggests difficulty of finding a maximum principle, passing through the transition stage (from vorticity layer to tubes). After turbulence settled in a developed stage with bunch of vortex tubes, the situation might be improved. Also noted is Rosen’s conjecture for the absence of a non self-similar blowup in a restricted sense. These are markedly open.

Here are some miscellaneous remarks. As long as inviscid fluids are concerned, the most fundamental formulation for ideal fluids is given by Hamiltonian methods [34, 35]. In such a variational formulation, there are important geometric quantities that characterise the flow fields, centering on the sectional curvature. This point of view has not been emphasised here. Because of its fundamental importance, it may be in order to ask how these ideas will help us understand possible singularity formation in fluids. Also, the problem of determining whether or not Lagrangian and Eulerian instabilities are equivalent is of interest. See [36] for details and for recent results.

Recently, there have also been applications of the field-theoristic methods in the basic study of fluid mechanics. For example, in [37] it was noted that "Most of the fluids examined so far arise out of underlying particles systems. (The sole exception is the Madlung-like construction based on an underlying filed – the Schrödinger field . . . )” See also [38]. At present such a construction is restricted to limited to flows. We may ask if there is any chance of generalising such an approach to flows with non-zero vorticity.

5. Appendix

It may be in order to describe Okamura’s theory briefly, see [17, 18] for details.

Prior to considering solutions of ordinary differential equations, let us consider a family of curves $\Omega$ by slightly generalising the problem. A curve $C \in \Omega$ is defined by $x(t) \in C^1$ on an interval $I$ of $t$. Let us define space-time $E$ by the set of all points $(x, t)$ on all curves $C$, that is,

$$E = \{ (x, t) \mid \text{points on } C \}.$$ 

We understand if there are two or more curves passing through a point $P \in E$, their $x(t)$ coincide. Then the directional fields are determined by $\Omega$ on $E$.

We introduce the function $D(P, Q)$. Consider two points in $E$ $P(x^p, t^p)$ and $Q(x^q, t^q)$ ($t^p \leq t^q$) and the subdivision of time

$$t^p \equiv t_0 \leq t_1 \leq \ldots \leq t_n \equiv t^q.$$
Figure 4. Schematic figure of the definition of Okamura function as a sum of jumps (heavy lines) across particle trajectories, after [19]. This is fictitious in the regular evolution.

Let the curves in $\Omega$ passing through $P$ and $Q$ be $C_0$ and $C_{n+1}$, respectively. We choose $n$ curves $C_k$ ($k = 1, \ldots, n$) between them arbitrarily. Let the arcs of $C_k$ associated with $t_{k-1} \leq t \leq t_k$ be $\overline{P_kQ_k}$, where $P_k$ and $Q_k$ are the left and right end points. We then form the following sum:

$$S = Q_0P_1 + Q_1P_2 + \ldots + Q_{n-1}P_n + Q_nP_{n+1},$$

(41)

where $Q_0 \equiv P$ and $P_{n+1} \equiv Q$, and $Q_kP_{k+1}$ is the distance at $t = t_k$. The physical meaning of $S$ is the sum of distances a fluid particle jumps instantaneously across $n$ particle paths in traveling from $P$ at $t = t^P$ and $Q$ at $t = t^Q$. For fixed $P, Q$, we define

$$D(P, Q) = \inf_{\text{subdivision, } n, C_k} S.$$ 

This has a number of properties

(i) $D(P, Q) \geq 0$.
(ii) $P$ and $Q$ ($t^P \leq t^Q$) lie on the same trajectory on $E \iff D(P, Q) = 0$
(iii) $D(P, Q)$ does not increase with $t$, when $Q$ moves along a particle trajectory.

It is important that not only $\Rightarrow$ but also $\Leftarrow$ holds in the second property (ii). We note
that we may alternatively define the function $D(P,Q)$ by
\[
\inf_{t \in P} \int_t^Q \sqrt{\frac{d}{dt} \left| u(x,t) \right|^2} \, dt.
\]

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